DOBRAKOV-DUNFORD AND BIRKHOFF-DUNFORD INTEGRAL WITH VECTOR MEASURE

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Abstract. Let X, Y are real Banach spaces. We study integration of Banach space valued functions with respect to Banach space valued measures. Our main result states that Birkhoff-Dunford integrability implies Mcshane-Dunford integrability. We also show that a function is measurable and Mcshane-Dunford integrable if and only if Dobrakov-Dunford integrable.

Keywords and phrases: Dobrakov-Dunford; Mc shane Dunford; Birkhoff-Dunford; Pettis integral

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1. Introduction

During the period of Banach space, the theory of integration of vector valued function with respect to vector measure was studied. (see [14] for overview). Bartle, Dobrakov methods are most known (see [11, 13]). Several authors have worked on this topic. More about this (see [14, 16, 18, 19]) the Dobrakov theory including most of the theory one common feature: the function are required to be measurable. Unfortunately we found in [4, 6] the Birkhoff and McShane integral do not required strong measurability. Many recent author has caught the attention here (see [7, 8, 10]) and the references given here. Both the integrals are defined as limits of sums of the form $\sum_i \mu(A_i)f(t_i)$, where (A_i) is a countable family of pairwise disjoint measurable sets and t_i's are points of the domain which are related to the A_i's in some way. It is natural to try to extend these integrals to the more general setting of vector-valued measures and our purpose here is to study such generalizations, which are obtained as follows: we will consider a vector measure μ with values in the Banach space $L(\mathcal{Z}, Y)$ of all bounded operators from $\mathcal{Z}$ to another Banach space Y. We replacing the sum $\sum_i \mu(A_i)f(t_i)$ the product by scalars $R \times \mathcal{Z} \to \mathcal{Z}$ with the natural bilinear map $L(\mathcal{Z}, Y) \times \mathcal{Z} \to Y$. Recalling S^*-

integral of Dobrakov which is derived from Kolmogorov's approach to integration theory, is the natural extension of the Birkhoff integral to the case of vector valued functions and vector valued measures. We consider vector measure μ with values in the Banach space $\mathcal{Z}$ with measure space$(\mathcal{Z}, \otimes, \mu)$ where $\otimes \subset \mathbb{R}^n$ or $(\mathbb{R})$.
In first section we discuss Dobrakov-Dunford Integral and relationship with Dunford Integral, McShane-Dunford Integral and its relatioship with Dunford Integral with relatioship between McShane-Dunford Integral and Denjoy-Dunford Integral. For Denjoy-Dunford see [21]. Relationship between Dobrakov-Dunford Integral with McShane-Dunford Integral also we study in this section. We study little about Birkhoff-Dunford Integral.
In section 2 we discuss Dobrakov-Pettis integral, McShane-Pettis integral and its relationship with Pettis integral. We discuss in weakly sequentially complete Banach space Dobrakov-Dunford integral and McShane-Dunford integral are equivalent to Dobrakov-Pettis and Mcshane-Pettis integral.
In [9], author discussed S^*-integrability implies McShane integrability in contexts in which the later notion is definable. They also show that a function is measurable and McShane integrable if and only if it is Dobrakov integrable (i.e. Bartle $*-$integrable. Finally, we obtain our main result states that Birkhoff-Dunford integrable function is McShane-Dunford integrable (and the respective Integrals coincide).

2. Dobrakov-Dunford, McShane-Dunford and Birkhoff-Dunford Integrals and Their properties

Recalling about Dunford integral and few Definition in the beginning of this section.

Dunford integral: A clever new point of view was injected into the theory of Lebesgue integration by Dunford (see [3]) who observed that if one normed the space of continuous functions by the condition $||f|| = \int |f(s)|ds$, the resulting linear normed space is not complete. If one completes this metric space in the metric space in the usual way, one gets exactly the Lebesgue integrable functions. By changing the norm of f to $||f|| = |\int |f(s)|^p ds|^{\frac{1}{p}}$, $p \geq 1$ the space L^p of functions for which $|f(s)|^p$ is integrable emerges.Adapting this technique, Dunford assumes a

completely additive set function a defined on a $\sigma-$field of subsets of $n-$dimensional Euclidean space, which includes the Borel measurable sets. Dunford extends his definition to a metric space, probably because he started with an integral on the space of continuous functions. Later he noticed that a more elegant approach is via the space of finite-valued measurable functions, since this does not involve any topological conditions on . The next step is then obvious: we take a general space , a class of (measurable subsets) S, which forms a (σ-field, a completely additive set function a on E to real numbers, and define an integral for the class of finite-valued measurable functions. The semi variation

$$\widehat{\mu}(A) = \sup\{||\sum_{i=1}^{n} \mu(A_i)(x_i)||\}$$

where the sup runs over all finite measurable partitions $(A_i)_{i=1}^n$ of Ω and all finite sequences $(x_i)_{i=1}^n$ in X. In general $\widehat{\mu}(A) \in [0, \infty]$. The most important case is when $\widehat{\mu}$ is continuous, that is $\widehat{\mu}(A_n)_n \to 0$ when $(A_n)_n \to \emptyset($ in $\Sigma)$.Equivalently by a result of Dobrakov $\widehat{\mu}$ has a control measure, that is there exists a positive finite measure λ on Σ such that $\lambda(A) \leq \widehat{\mu}(A)$ and $\widehat{\mu}(A) \to 0$ when $\lambda(A) \to 0$. In this case $\widehat{\mu}$ is bounded. Note that the condition $\widehat{\mu}(\otimes) < \infty$ does not implies in general, that $\widehat{\mu}$ is continuous. However, if Y does not have a copy of C_0, these conditions are equivalent. When the sub measure $\widehat{\mu}$ is $\sigma-$finite and locally continuous, by this we mean that $\otimes$ can be write as $\bigcup_n \otimes_n$ where $\widehat{\mu}(\otimes) < \infty$ for every n and each restriction $\widehat{\mu} \setminus \Sigma_n,\ \Sigma_n = \{A \cap \Omega_n\ : A \in \Sigma\}$ is continuous, these two conditions are equivalent to have a $\sigma-$finite control measure for $\widehat{\mu}$.

Definition 2.1. [5]

(1) *A function $h : \otimes \to Y$ is scalar measurability if $< y^*, h >$ is measurable for each $y^* \in Y^*$.*
(2) *The function $h : \otimes \to X$ is strongly measurable or Bochner measurable if only if h is pointwise limit(a.e.) of a sequence of measurable simple functions*

In general Scalar measurability and strongly measurability are not equivalent except for particular case of Banach spaces. Through out the article we assume strongly measurability is equivalent to scalar measurability that is our measurable function are almost separably valued.

Definition 2.2. (1) *A function $h : \otimes \to \mathcal{Z}$ is called measurable if there exists a sequence of simple functions converging pointwise to f.*

(2) *A function $h : \Omega \to \mathcal{Z}$ is Dobrakov integrable with respect to μ, if it is measurable and there is a sequence of simple $\mathcal{Z}-$valued functions (h_n) converging to h $\widehat{\mu}-$ almost everywhere such that for every $E \in \Sigma$, there exists $\lim\limits_{n} \int_E h_n d\mu_E$, for the norm topology of Y; is called Dobrakov integral. Dobrakov-integral of h with respect to μ is*

$$D \int_{\otimes} h d\mu = \lim_{n} \int_{\Omega} h_n d\mu$$

where $\widehat{\mu}$ is semivariation of μ.

(3) *A function $h : \Omega \to \mathcal{Z}$ is S^*- integrable with respect to μ with $S^* \int_{\Omega} hd\mu \in \mathcal{Z}$ if for every $\epsilon > 0$ there exists $A_0 \in P(\Omega)$ such that $A = (A_n)_n \in P(\Omega, h)$ for every $A \geq A_0$ and*

$$|| \sum_{n=1}^{\infty} h(w)\mu(A_n) - S^* \int_{\Omega} h\mu|| \leq \epsilon$$

with uniqueness of $S^* \int_{\Omega} hd\mu$.

We state our definition of Dobrakov-Dunford, Mcshane-Dunford and Birkhoff-Dunford integral as:

Definition 2.3. *A function $h : \Omega \to \mathcal{Z}$ is called Dobrakov-Dunford integrable on $\otimes$ if for each $y^* \in \mathcal{Z}^*$ the function $y^* f$ is Dobrakov integrableon $\otimes$ and if for every subset E of $\otimes$ there exists a vector $y_E^{**} \in \mathcal{Z}^{**}$ such that*

$$y_E^{**}(y^*) = \int_E y^* h$$

or $y^* \in \mathcal{Z}^*$. We write

$$y_E^{**} = DD \int_E h = H(E).$$

Definition 2.4. *A function $f : \Omega \to X$ is McShane-Dunford integrable on $\otimes$ if for each x^* in X^* the function $x^* f$ is McShane integrable on $\otimes$ and for every subsetE of Ω, there is a vector $x_{E^*} \in X^{**}$ such that*

$$x_E^*(x^*) = M \int_E x^* f \ \forall x^* \in X^*$$

We write

$$x_E^{**} = MD \int_E f$$

Definition 2.5. *$f:\Omega\to X$ is Birkhoff-Dunford with respect to μ with $S^*\int_\Omega f d\mu\in X$ if x^*f is Birkhoff that is for $\epsilon>0$ there exists $A_0\in P(\Omega)$ such that $A=(A_n)_n\in P(\Omega,f)$ for every $A\geq A_0$ and*

$$||\sum_{n=1}^{\infty}f(w)\mu(A_n)-S^*\int_\Omega x^*fd\mu||\leq\epsilon$$

The function h is integrable in one of the above sense (Definition (2.3 -2.5) on the set $E\subset\Omega$ if the function $fch(E)$ is integrable in that sense on Ω.

Proposition 2.6. *If a function $f:\Omega\to X$ is Dobrakov-Dunford on $\otimes$, if and only ifx^*f is Dobrakov integrable on $\otimes$, for each $x^*\in X^*$*

Proof. Let h is Dobrakov-Dunford integrable on Ω, for every $x^*\in X^*$, by the definition x^*f is Dobrakov integrable on Ω.
Conversely, let x^*f is Dobrakov integrable on Ω. Then

$$D\int_\Omega fx^*d\mu=\lim_n\int_\Omega f_nx^*d\mu$$

That is

$$D\int_\Omega fd\mu=\lim_n\int_\Omega f_nd\mu$$

So, h is Dobrakov-integrable. □

Proposition 2.7. *If a function $f:\Omega\to X$ is Dobrakov-Dunford on $\otimes$, then each perfect set in $\otimes$ contain a portion of which f is Dunford integrable.*

Proof. Since $f:\Omega\to X$ is Dobrakov-Dunford integral on Ω, then for each $x^*\in X^*$, x^*f is Dobrakov integrable on Ω. As [20] each perfect set in $\otimes$ contains a portion on which x^*f is Lebesgue integrable. So, h is Dunford integral. □

Theorem 2.8. *If a function $f:\Omega\to X$ is Dobrakov-Dunford on $\otimes$, then there exists a sequence $\{X_k\}$ of closed sets, $\bigcup_{k=1}^\infty X_k=\Omega$, then h is Dunford integrable on each X_k.*

Proposition 2.9. *If $f:\Omega\to X$ is McShane-Dunford integrable on Ω if and only if f is Denjoy-Dunford integrable on Ω, for all $x^*\in X^*$.*

Proof. If f is McShane-Dunford integrable on Ω, then by definition x^*f is McShane integrable on Ω. Using Fleming theorem and Th 2.2 of [12] , h is Denjoy-Dunford

integrable on Ω.

Conversely, let h is Denjoy-Dunford integrable on Ω, then x^*f is Denjoy integrable and $\int_E x^*f = M\int_E x^*f$. Using Lemma 1.5 of [12] that f is Denjoy-Dunford on $\otimes$ and for every subsets E of $\otimes$ there is a vector $x^*_E \in X^{**}$ such that

$$x^{**}_E(x^*) = DD\int_E x^*f \;\forall x^* \in X^*$$

That is

$$x^{**}_E(x^*) = MD\int_E x^*f \;\forall x^* \in X^*$$

Hence h is McShane-Dunford integrable on Ω. □

Theorem 2.10. *If a function $f : \Omega \to X$ is McShane-Dunford on $\otimes$, if and only ifx$^*f$ is McShane integrable on $\otimes$, for each $x^* \in X^*$*

Proof. If a function $f : \Omega \to X$ is McShane-Dunford on $\otimes$, then x^*f is McShane integrable on Ω, for each $x^* \in X^*$.

Conversely let, x^*f is McShane integrable. Fremlin's theorem saying x^*f is Henstock and Pettis integral. So, h is Denjoy-Dunford integrable. That is h is McShane Dunford integrable. □

Theorem 2.11. *If a function $f : \Omega \to X$ is McShane-Dunford on $\otimes$, then there exists a sequence $\{X_k\}$ of closed sets, $\bigcup_{k=1}^{\infty} X_k = \Omega$, then h is Dunford integrable on each X_k.*

Proof. Since $f : \Omega \to X$ is McShane-Dunford integrable on Ω, then f is Denjoy-Dunford . That is for each $x^* \in X^*$, x^*f is Denjoy integrable. Now each perfect set in $\otimes$ contains a portion on which x^*f is lebesgue measurable. So, f is Dunford integrable on a portion. □

Theorem 2.12. *If the function $f : \Omega \to X$ is McShane-Dunford integrable on $\otimes$ then there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\cup_{k=1}^{\infty} X_k = \Omega$, f is Dunford integrable on each X_k and*

$$\lim_{k\to\infty} D\int_{X_k \cap E_0} = MD\int_{E_0} f(t)dt$$

weakly uniformly on Ω.

Theorem 2.13. *If $f : \Omega \to X$ is Birkhoff-Dunford integral if and only if x^*f is Birkhoff integrable.*

Proof. If $f : \Omega \to X$ is Birkhoff-Dunford integral then x^*f is Birkhoff integrable. Conversely, let x^*f is Birkhoff integrable. Using Theorem 3.7 of [9] x^*f is McShane integrable with respect to μ and

$$S^* \int_{\Omega} x^* f d\mu = M \int_{\Omega} x^* f d\mu$$

So, f is McShane-Dunford integrable and hence f is Birkhoff-Dunford integrable. □

Proposition 2.14. *If a function $f : \Omega \to X$ is Birkhoff-Dunford on $\otimes$, then each perfect set in $\otimes$ contain a portion of which f is Dunford integrable.*

Proof. Since $f : \Omega \to X$ is Birkhoff-Dunford integral on Ω, then for each $x^* \in X^*$, x^*f is Birkhoff integrable on Ω. As [20] each perfect set in $\otimes$ contains a portion on which x^*f is Lebesgue integrable. So, h is Dunford integral. □

3. Dobrakov-Pettis, McShane-Pettis and Birkhoff-Pettis integral

In this section we discuss Dobrakov, McShane and Birkhoff integral with weak integral called Pettis integral. For detailed of Pettis integral see [17].

Definition 3.1.

(1) *A function $f : \Omega \to X$ is called Dobrakov-Pettis integral if h is Dobrakov-Dunford on $\otimes$ and $x_E^{**} \in X$ for every subsets E in Ω, we write*

$$x_{E_0}^{**} = DP \int_{E_0} f$$

(2) *A function $f : \Omega \to X$ is called McShane-Pettis integral if h is McShane-Dunford on $\otimes$ and $x_E^{**} \in X$ for every subsets E in Ω, we write*

$$x_{E_0}^{**} = MP \int_{E_0} f$$

(3) *A function $f : \Omega \to X$ is called S^*-Pettis integral if h is S^*-Dunford on $\otimes$ and $x_E^{**} \in X$ for every subsets E in Ω, we write*

$$x_{E_0}^{**} = BP \int_{E_0} f$$

Theorem 3.2.

(1) *Suppose that $\mathcal{Z}$ contains no copy C_0 and $h : \Omega \to \mathcal{Z}$ is Dobrakov-Pettis integrable on Ω, then each perfect set in $\otimes$ contains a portion on which f is Pettis integrable.*

(2) *Suppose that* $\mathcal{Z}$ *contains no copy* C_0 *and* $h : \Omega \to \mathcal{Z}$ *is McShane-Pettis integrable on* Ω, *then each perfect set in* $\otimes$ *contains a portion on which* f *is Pettis integrable.*

(3) *Suppose that* $\mathcal{Z}$ *contains no copy* C_0 *and* $h : \Omega \to \mathcal{Z}$ *is Birkhoff-Pettis integrable on* Ω, *then each perfect set in* $\otimes$ *contains a portion on which* f *is Pettis integrable.*

Proof. For (1) Since $h : \Omega \to \mathcal{Z}$ is Dobrakov-Pettis integrable on Ω, then h is Dobrakov-Dunford integral. This is means each perfect set in $\otimes$ contains a portion of which f is Dunford integral. As $\mathcal{Z}$ contains no copy of C_0, by Theorem 7 of [17] h is Pettis integrable.
The proof of (2) and (3) are similar to (1) with their definitions. □

Theorem 3.3. (1) *Suppose* $\mathcal{Z}$ *contains no copy of* C_0 *and* $f : \Omega \to X$ *is measurable. If the function* f *is Dobrakov-Pettis integrable on* Ω, *then there exists a sequence* $\{X_k\}$ *of closed sets with* $X_k \uparrow \Omega$ *such that for each* $x^* \in X^*$, f *is Pettis integrable on each* X_k *and*

$$\lim_{k\to\infty} Pettis \int_{X_k} f = DP \int_E f$$

weakly.

(2) *Suppose* $\mathcal{Z}$ *contains no copy of* C_0 *and* $f : \Omega \to X$ *is measurable. If the function* f *is McShane-Pettis integrable on* Ω, *then there exists a sequence* $\{X_k\}$ *of closed sets with* $X_k \uparrow \Omega$ *such that for each* $x^* \in X^*$, f *is Pettis integrable on each* X_k *and*

$$\lim_{k\to\infty} Pettis \int_{X_k} f = MP \int_E f$$

weakly.

(3) *Suppose* $\mathcal{Z}$ *contains no copy of* C_0 *and* $f : \Omega \to X$ *is measurable. If the function* f *is Birkhoff-Pettis integrable on* Ω, *then there exists a sequence* $\{X_k\}$ *of closed sets with* $X_k \uparrow \Omega$ *such that for each* $x^* \in X^*$, f *is Pettis integrable on each* X_k *and*

$$\lim_{k\to\infty} Pettis \int_{X_k} f = BP \int_E f$$

weakly.

Proof. Proof are very similar to the Theorem 2.7 of [12]. □

Corollary 3.4. *Suppose $\mathcal{Z}$ contains no copy of C_0. If the function $f : \Omega \to X$ is Dobrakov-Pettis integrable on $\otimes$ then there exists a sequence $\{X_k\}$ of closed sets, $\cup_{k=1}^{\infty} X_k = \Omega$ and f is Pettis integrable on each X_k.*

Remark 3.5. *The above corollary is true for McShane-Pettis and Birkhoff-Pettis integral also.*

We discuss now about equivalent condition of Dobrakov-Dunford and Dobrakov-Pettis integral, also equivalent condition of McShane-Dunford and McShane-Pettis integral for the case of measurable function. When we discuss Pettis integrable functions, the space C_0 create difficulties.If $C_0 \subset X$ isomorphically then there are $X-$valued scalarly integrable functions that are not Pettis integral. This is the reason we consider $\mathcal{Z}$ is weakly sequentially complete and trying to find our results.

Lemma 3.6.
(1) *Suppose $\mathcal{Z}$ is weakly sequentially complete, then a function $f : E \to X$ is Dobrakov-Pettis integrable only if for each closed set $E_0 \subset E$, there exists a portion $P = E_0 \cup E$ of E on which h is Pettis integrable.*
(2) *Suppose $\mathcal{Z}$ is weakly sequentially complete, then a function $f : E \to X$ is Mcshane-Pettis integrable only if for each closed set $E_0 \subset E$, there exists a portion $P = E_0 \cup E$ of E on which h is Pettis integrable.*

Proof. For (1) As for each closed set $E_0 \subset E$, there exists a portion $P = E_0 \cup E$ of E on which h is Pettis integrable. Then there exists a sequence $f_n : E \to X$ of $X-$valued simple function such that for each$x^* \in X^*$, $x^* f = \lim_n x^* f_n$, $\mu-$ a.e. That is there is a sequence of simple $X-$ valued function $\{f_n\}$ converging to f $\widehat{\mu}-$a.e. such that

$$\int_E x^* f = \lim_n \int_E x^* f_n$$

So,

$$\int_E f = \lim_n \int_E f_n$$

This gives h is Dobrakov integrable on P. Hence $f : E \to X$ is Dobrakov-Pettis integral.

For (2) As for each closed set $E_0 \subset E$, there exists a portion $P = E_0 \cup E$ of E on which h is Pettis integrable. That is h is McShane integrable on P. □

Theorem 3.7. (1) *Suppose $\mathcal{Z}$ is weakly sequentially complete and $f : E \to X$ is measurable. If h is Dobrakov-Dunford on $E \subset \Omega$ then h is Dobrakov-Pettis on E.*

(2) *Suppose $\mathcal{Z}$ is weakly sequentially complete and $f : E \to X$ is measurable. If h is McShane-Dunford on $E \subset \Omega$ then h is McShane-Pettis on E.*

Proof. For (1) Since h is Dobrakov-Dunford on E, then for each closed set $E_0 \subset E$ there exists a portion $P = E \cup E_0$ on which h is Dunford integrable. Here $\mathcal{Z}$ is weakly sequentially complete therefore $\mathcal{Z}$ contains no copy of C_0, so h is Pettis integrable on P, hence f is Dobrakov-Pettis on E. □

Corollary 3.8. *Assume $\mathcal{Z}$ is weakly sequentially complete. If $\mathcal{Z}$ is separable and function are Dobrakov-Dunford and Mcshane-Dunford integrable on $E \subset \Omega$, then function are Dobrakov-Pettis and Mcshane-Pettis integrable, respectively.*

Remark 3.9. *If $\mathcal{Z}$ is Schur space then $\mathcal{Z}$ is weakly sequentially complete, so Dobrakov-Dunford and McShane-Dunford integrable on $\mathcal{Z}$ are Dobrakov-Pettis and McShane-Pettis integrable, respectively.*

In [9] author proved that Birkhoff integrable if and only if McShane integrable with respect to μ. In the following theorems we prove the results of [9] are true if we Dunford integral is clubbed with Birkhoff and McShane integral. Similarlly for Dobrakov integral also we have discussed.

Theorem 3.10. *If $f : \Omega \to X$ is Birkhoff-Dunford integrable with respect to μ if and only if f is McShane-Dunford integrable with respect to μ and*

$$S_D^* \int_\Omega f d\mu = MD \int_\Omega f d\mu.$$

Proof. Let $f : \Omega \to X$ is Birkhoff-Dunford integral then x^*f is Birkhoff integral and using Theorem 3.7 of [9] h is McShane-Dunford integrable.
Conversely, let $f : \Omega \to X$ is McShane-Dunford integral then x^*f is McShane integrable. Using Corollary 4.2 of [1], x^*f is Birkhoff integral and hence f is Birkhoff-Dunford integral. □

Theorem 3.11. *A measurable function $f : \Omega \to X$ is McShane-Dunford integrable if only if h is Dobrakov-Dunford integral.*

Proof. If h is Dobrakov-Dunford integrable with respect to μ. So, $x^(f$ is Dobrakov integral also x^*f is measurable so using Theorem 3.8 of [9] x^*f is McShane integrable and hence h is McShane-Dunford integral.
Converse part we can proof using Theorem 3.8 of [9] and using McShane-Dunford's property. □

4. Conclusion

In this article we introduced Dobrakov-Dunford, McShane-Dunford and Birkhoff-Dunford integral. We discussed their properties to prove our main results Birkhoff-Dunford integrability implies McShane-Dunford integrability and McShane-Dunford integrable if and only if Dobrakov-Dunford integrable.
Ethical approval: This article does not contain any studies with human participants or animals performed by the authors.

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